Quasinormal modes in the large angular momentum limit: an inverse multipolar expansion analysis

A Chrysostomou[1] **and A S Cornell**[2]

Department of Physics, University of Johannesburg, PO Box 524, Auckland Park 2006, South Africa.

E-mail: [1]`annachrys97@gmail.com`, [2]`acornell@uj.ac.za`

Abstract. The quasinormal modes (QNMs) of a black hole (BH) may be identified as a class of damped, classical oscillations in spacetime, emergent as part of the late-stage response to a perturbation of the BH. In a recent paper, we utilised the inverse multipolar expansion method proposed by Dolan and Ottewill to investigate the quasinormal frequencies of 4D Schwarzschild, Reissner-Nordström, and Schwarzschild de Sitter BHs within the eikonal limit for fields of spin $s = \{0, 1/2, 1, 3/2, 2\}$. Here, we extend this method to the calculation of the radial component of the QNM wavefunctions within the Schwarzschild BH spacetime for each of these fields, investigating specifically the behaviour of these wavefunctions within the large-ℓ regime.

1. Introduction

The regular detection of binary black hole (BH) mergers has provided us with a wealth of data against which we may test our extant models of gravitational-waves (GWs) and their BH sources [1]. The demonstrable consensus between GW data and modelling [2] validates our understanding of BH mergers as three-phase events successfully described using (i) the post-Newtonian approximation, (ii) numerical relativity, and (iii) BH perturbation theory. These phases are, respectively, (i) inspiral, a long adiabatic stage as the orbit shrinks and GW emission increases; (ii) merger, a violent coalescence of these compact bodies into a single BH such that GW emission peaks; (iii) ringdown, where the final BH emits damped GWs as it relaxes into a stationary state [3, 4].

The damped oscillations in spacetime from which ringdown is comprised are known as the quasinormal modes (QNMs). Their corresponding quasinormal frequencies (QNFs) may be decomposed into a real and imaginary part, where the former represents the physical oscillation frequency and the latter is related to the decay timescale of the BH's perturbation. Unlike the oscillations within the inspiral and merger phases, the QNFs are independent of the initial conditions; they depend exclusively on the characteristics of the final BH [5]. This, as well as their relationship to gauge-gravity duality [6] and possibly even BH area quantisation [7], had led to immense interest in QNFs within the field of BH physics and beyond.

However, the computation of BH QNFs is wrought with technical difficulty due to the inherently dissipative nature of the BH system. This is a consequence of the boundary conditions thereof: in BH spacetimes with a cosmological constant $\Lambda \geq 0$, energy is irrevocably lost at the BH event horizon and at spatial infinity (or at the cosmological horizon of the de Sitter (dS) cases denoted by $\Lambda > 0$). As such, QNMs do not form a complete set in these contexts (see Ref. [8] for further discussion). Finally, QNMs may be shown to be non-normalisable in asymptotically-flat and dS spacetimes (but can be normalisable in anti-de Sitter (AdS) spacetimes) [9].

To contend with these technical challenges, a number of methods have been developed within the QNM literature (concisely reviewed in Refs. [5, 9]). As explained in Ref. [10], the computational method applied to solve for QNFs must be chosen carefully such that the specifics of the BH spacetime and wave equation dependencies are accommodated, for a certain approach may fail under conditions where another proves more accurate. Here, we utilise a method derived directly from the BH context: the inverse multipolar expansion method put forth by Dolan and Ottewill in Ref. [11]. The "Dolan-Ottewill" method involves the introduction of a novel ansatz for the QNM wave equation (described in section 2) constructed from the circular null geodesics of a spherically-symmetric BH in non-AdS spacetime. The QNF can then be solved for through an iterative process, and emerges as an expansion in inverse powers of the multipolar number (ℓ). Consequently, the method is explicitly useful in exploring large-ℓ asymptotics [12].

In this work, we expand upon our paper, Ref. [13], by explaining the application of the Dolan-Ottewill method in greater detail. We then demonstrate how the method can be extended to the computation of QNM wavefunctions of spin $s = \{0, 1/2, 1, 3/2, 2\}$ for the 4D Schwarzschild BH. Finally, we impose the large-ℓ limit on the wavefunctions and assess their behaviour therein.

2. QNM effective potentials in the 4D Schwarzschild BH spacetime

A non-rotating, spherically-symmetric, 4D BH may be described in its most general terms by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \ . \tag{1}$$

The Schwarzschild metric function is given by $f(r) = 1 - r_{_H}/r$, with the event horizon located at $r_{_H} = 2M$ (setting $\hbar = G = c = 1$). To analyse the perturbation of such a BH, we may decompose the spacetime into a background metric $\overset{BG}{g_{\mu\nu}}$ and a small perturbing term $h_{\mu\nu}$,

$$g'_{\mu\nu} = \overset{BG}{g_{\mu\nu}} + h_{\mu\nu} \ , \quad h_{\mu\nu} \ll \overset{BG}{g_{\mu\nu}} \ . \tag{2}$$

We then solve the consequent system of linear differential equations that satisfy Einstein's vacuum field equations [14]. If we consider a scalar test field Φ as a perturbing field on this linearised gravitational background, it may be shown that $h_{\mu\nu}$ and Φ decouple such that the metric perturbations described by $h_{\mu\nu}$ can be set to zero (see Ref. [5] for details). With the tortoise coordinate $dx/dr = 1/f(r)$, we can isolate the radial behaviour of the QNM as

$$\frac{d^2}{dx^2}\Phi(x) + \left[\omega^2 - V_{eff}(r)\right]\Phi(x) = 0 \ , \tag{3}$$

where ω represents the QNF. The effective QNM potentials associated with fields of integer spin s within the Schwarzschild BH spacetime can be concisely expressed through

$$V_{eff}(r) = \frac{f(r)}{r^2}\left[\ell(\ell+1) + \frac{2M(1-s^2)}{r}\right] \ , \tag{4}$$

$$s = \begin{cases} 0 \ , & \text{scalar perturbations} & \Rightarrow (1-s^2) = \ \ 1 \\ 1 \ , & \text{electromagnetic perturbations} & \Rightarrow (1-s^2) = \ \ 0 \\ 2 \ , & \text{gravitational perturbations: vector-mode} & \Rightarrow (1-s^2) = -3 \ . \end{cases}$$

On the basis of the isospectrality of the spin-2 scalar- and vector-modes [15], here we shall consider only the vector-modes. For half-integer fields, the QNM potentials can be cast as isospectral supersymmetric partners through

$$V_{eff} = \pm f(r)\frac{d}{dr}W + W^2 \ , \quad W = \frac{\sqrt{f(r)}}{r}\kappa z \ , \tag{5}$$

where the superpotential W is dependent on $\kappa \equiv j + 1/2$, the spinor eigenvalue on the 2-sphere, and

$$\kappa z = \begin{cases} \kappa \ , & \kappa = \ell + 1 \ , \ \text{Dirac perturbations} \qquad s = 1/2 \\ \kappa \dfrac{\kappa^2 - 1}{\kappa^2 - f(r)} \ , & \kappa = \ell + 2 \ , \ \text{Rarita-Schwinger perturbations} \qquad s = 3/2 \ . \end{cases} \tag{6}$$

respectively [19]. Note that there exists a distinction between the definition of κ: while $j = \ell \pm 1/2$ for Dirac cases [16], $j = \ell \pm 3/2$ for Rarita-Schwinger fields [17, 18]. Here, as in Ref. [13], we define the spinor eigenvalue on the 2-sphere in terms of ℓ (where ℓ is an integer).

3. The Dolan-Ottewill method: Schwarzschild QNFs

The primary objective of the multipolar expansion method is to express the QNF as a linear expansion in inverse mutlipolar numbers,

$$\omega = \sum_{k=-1} \omega_k L^{-k} \ , \qquad L = \ell + 1/2 \ , \tag{7}$$

where each ω_k of the summation is computed iteratively. The procedure begins by defining the novel ansatz,

$$\Phi(r) = e^{i\omega z(x)} v(r) \ , \qquad z(x) = \int^x \rho(r) dx \ , \tag{8}$$

where the ansatz must adhere to the non-AdS boundary conditions,

$$f(r) \to 0 \ , \ \ b_c k_c(r) \to -1 \quad \text{as} \quad x \to -\infty \ , \tag{9}$$

$$f(r)/r^2 \to 0 \ , \ \ b_c k_c(r) \to +1 \quad \text{as} \quad x \to +\infty \ . \tag{10}$$

These convey the fact that the event horizon is encountered as $x \to -\infty$ and an asymptotically-flat region or cosmological horizon is approached as $x \to +\infty$. The ansatz may be substituted into equation (3) to obtain

$$f(r) \frac{d}{dr} \left(f(r) \frac{dv}{dr} \right) + 2i\omega\rho(r) \frac{dv}{dr} + \left[i\omega f(r) \frac{d\rho}{dr} + \left(1 - \rho(r)^2\right) \omega^2 - V(r) \right] v(r) = 0 \ . \tag{11}$$

Though not obvious by inspection, this equation is simpler to solve than equation (3) as it lends itself with greater ease to the iterative procedure of Dolan and Ottewill.

We may then define $\rho = b_c k_c(r)$. The ansatz therefore depends on the impact parameter $b = r/\sqrt{f(r)}$ and the newly-defined function $k(r) = b^{-2} - f(r)/r^2$, both of which are evaluated at the critical orbit $r = r_c$. This r_c can be derived using the metric function, $r_c = 2f(r)/\partial_r f(r)|_{r=r_c}$. Finally, we may define the function $v(r)$ as a further expansion in L^{-k}, namely

$$v(r) = \exp \left\{ \sum_{k=0} S_k(r) L^{-k} \right\} \ . \tag{12}$$

For the Schwarzschild BH spacetime, where we opt to set $M = 1$, the components of the ansatz are given by $r_c = 3$, $b_c = \sqrt{27}$, and thus $\rho(r) = (1 - 3/r)(1 + 6/r)^{1/2}$. We substitute these into equation (11), in conjunctio with equations (7) and (12) expanded to our order of choice. Here, we select $\mathcal{O}\left(L^{-6}\right)$. Lastly, we substitute the potential corresponding to our perturbing field of interest.

The procedure then involves collecting powers of L, setting the coefficient to zero, and solving for ω_k and $S'_k(r)$ for monotonically increasing values of k. The process can become fairly

Table 1: The inverse multipolar expansions for the effective QNFs of spin s at order $\mathcal{O}(L^{-k})$, from Ref. [13]. Odd (even) values of k correspond to real (imaginary) expansion terms. While the lowest-order terms remain constant for each field, we find that for each $k \geq 0$, the magnitude of the L^{-k} coefficients increases with s. Note that for QNFs of half-integer spin, we parametrise the multipolar number as $L \to \bar{L} = \kappa$, using the values of κ specified in equation (6).

s	$b_c \sum_{k=-1}^{6} \omega_k L^{-k}$
	Perturbations of integer spin
0	$L - \frac{i}{2} + \frac{7}{216L} - \frac{137}{7776L^2}i + \frac{2615}{1259712L^3} + \frac{590983}{362797056L^4}i - \frac{42573661}{39182082048L^5} + \frac{11084613257}{8463329722368L^6}i$
1	$L - \frac{i}{2} - \frac{65}{216L} + \frac{295}{7776L^2}i - \frac{35617}{1259712L^3} + \frac{3374791}{362797056L^4}i - \frac{342889693}{39182082048L^5} + \frac{74076561065}{8463329722368L^6}i$
2	$L - \frac{i}{2} - \frac{281}{216L} + \frac{1591}{7776L^2}i - \frac{710185}{1259712L^3} + \frac{92347783}{362797056L^4}i - \frac{7827932509}{39182082048L^5} - \frac{481407154423}{8463329722368L^6}i$
	Perturbations of half-integer spin
1/2	$\bar{L} - \frac{i}{2} - \frac{11}{216\bar{L}} - \frac{29}{7776\bar{L}^2}i + \frac{1805}{1259712\bar{L}^3} + \frac{27223}{362797056\bar{L}^4}i + \frac{23015171}{39182082048\bar{L}^5} - \frac{6431354863}{8463329722368\bar{L}^6}i$
3/2	$\bar{L} - \frac{i}{2} - \frac{155}{216\bar{L}} + \frac{835}{7776\bar{L}^2}i - \frac{214627}{1259712\bar{L}^3} + \frac{25750231}{362797056\bar{L}^4}i - \frac{2525971453}{39182082048\bar{L}^5} + \frac{292606736465}{8463329722368\bar{L}^6}i$

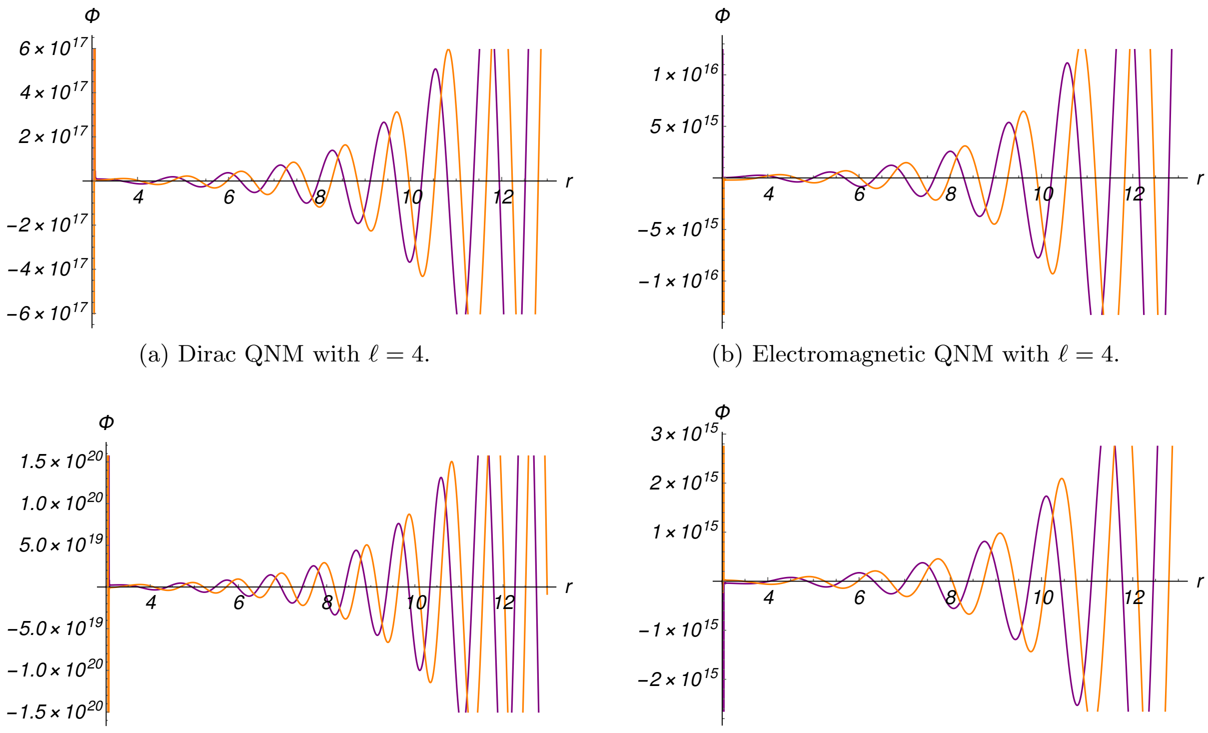
straightforward with the aid of the `Coefficient` and `Solve` functions within the MATHEMATICA environment. In this manner, we may compute the ω_k terms of equation (7), evaluate them at $r = r_c$, and express the QNF through the L^{-k}-dependent expansions provided in table 1. The lowest order terms are as follows:

$$
\begin{aligned}
L^2: &\quad 27\omega_{-1}^2 - 1 = 0 \quad \Rightarrow \omega_{-1} = \pm\frac{1}{\sqrt{27}}\,; \\
L^1: &\quad 2i\omega_{-1}\left(1+\frac{6}{r}\right)^{1/2}\left(1-\frac{3}{r}\right)S_0' + \frac{54\omega_{-1}\omega_0}{r^2} + \frac{27i\omega}{r^3}\left(1+\frac{6}{r}\right)^{-1/2} = 0 \\
&\qquad\qquad \Rightarrow \;\; \omega_0 \;= -\frac{i}{2\sqrt{27}} \\
&\qquad\qquad \Rightarrow S_0'(r) = \frac{\sqrt{27}}{r(r+6)(r-3)}\left[\left(1+\frac{6}{r}\right)^{1/2} - \frac{\sqrt{27}}{r}\right]\,.
\end{aligned}
$$

Thus, for every ω_k ($k \geq 0$) of table 1, we solve also for a $S_k'(r)$ term. These latter expressions are more complicated, and become undefined if $r = r_c$ is naively imposed. However, through integrating the $S_k(r)$ derivatives, we obtain the necessary terms to define $v(r)$. We may then substitute these various components of the ansatz into equation (8). The subsequent functions are plotted in figure 1 for spin $s = \{1/2, 1, 3/2, 2\}$, to order $\mathcal{O}(L^{-6})$ for $\ell = 4$.

Certain behaviours are consistent for all radial wavefunctions explored here, such as the divergence at the boundaries and the shift of $\pi/2$ between real and imaginary components. We observe, however, a decrease (an increase) in the amplitude for increasing s for integer (half-integer) cases. Irrespective of the spin, an increase in ℓ corresponds to an increase in the amplitude and a decrease in the wavelength of these radial profiles, as demonstrated in figure 2. Furthermore, there is notable shifting from one wavefunction to the next, with Dirac and gravitational wavefunctions shifted ahead of their Rarita-Schwinger and electromagnetic counterparts.

Despite these features, it was noted in Ref. [11] that the radial profile does not offer much in terms of physical insight. It is important to remember that our starting point of equation



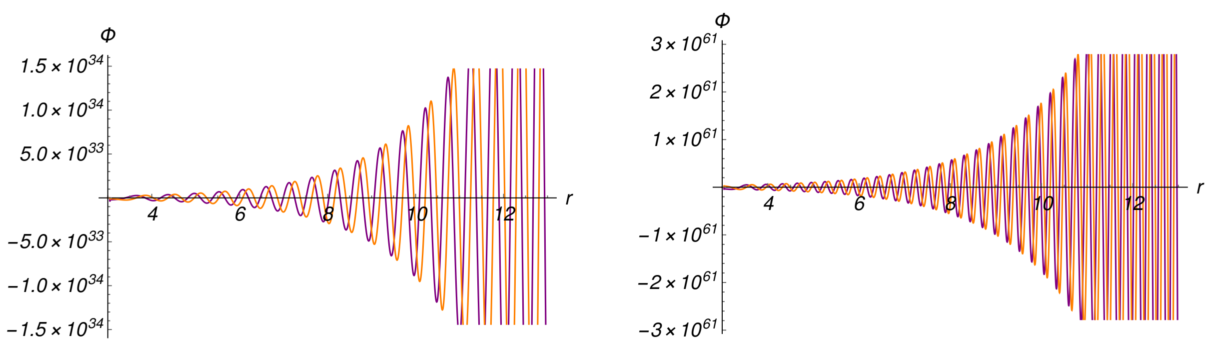
(a) Dirac QNM with $\ell = 4$.

(b) Electromagnetic QNM with $\ell = 4$.

(c) Rarita-Schwinger QNM with $\ell = 4$.

(d) Gravitational QNM with $\ell = 4$.

Figure 1: The radial wavefunctions for QNMs of spin $s = \{1/2, 1, 3/2, 2\}$ for Schwarzschild BHs with $\ell = 4$. Real and imaginary components are denoted by purple and orange, respectively.



(a) Dirac QNM with $\ell = 10$.

(b) Dirac QNM with $\ell = 20$.

Figure 2: The radial wavefunctions for QNMs of spin $s = 1/2$ for Schwarzschild BHs with $\ell = 10$ and $\ell = 20$. Real and imaginary components are denoted by purple and orange, respectively.

(3) represents the isolated radial behaviour. However, understanding the radial component is a necessary first step into calculating the full waveform in any 4D Schwarzschild application.

4. Conclusions

The Dolan-Ottewill method is an efficient, economical method that allows for the computation of QNFs to high orders in L^{-k} with relative ease. As a physically-motived method, it is easy to match the method with the appropriate context, and therefore avoid the limitations discussed in Ref. [10]. An especially interesting aspect of the method is its flexibility: here, we have demonstrated the simple means by which it can be extended to the computation of radial QNM wavefunctions; in Ref. [13], we explored the various non-rotating, spherically-symmetric BH spacetimes to which the Dolan-Ottewill method may also be applied. Since focus in the literature has been concentrated on the development of computational methods for QNFs [5, 9], the Dolan-Ottewill method becomes especially valuable for its use in the calculation of the relatively underexplored QNM wavefunctions.

Further extensions, however, are highly desirable *viz.* through an incorporation of temporal and angular components into our assessment of the QNM wavefunction. Studies into physically relevant aspects of the QNM contribution through the use of the Dolan-Ottewill method is already underway: in Ref. [12], Dolan and Ottewill combined their expansion method with a WKB analysis in order to compute the "QNM excitation coefficient" from the residues of the poles in the Green function for the scalar field within the 4D Schwarzschild context. This aids in the establishment of a more complete description for the QNM response in the wake of a BH perturbation. To explore these computations beyond the scalar field and Schwarzschild BH spacetime would be interesting.

Moreover, an investigation into the application of the Dolan-Ottewill method to higher-dimensional BHs may be possible. This was demonstrated in Ref. [11], where Dolan and Ottewill computed the lowest terms in the QNF expansion for the gravitational perturbations of a d-dimensional Schwarzschild BH. Whether this can be extended to AdS BH spacetimes remains an open question. Rotating BH spacetimes, however, may be accommodated. This was suggested in Ref. [20], where an additional expansion was introduced via the angular eigenvalue. We reserve exploration into these various avenues for future works.

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